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Asymptotic theory of high-aspect-ratio arched wings in steady incompressible flow

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Asymptotic theory of high-aspect-ratio wings in steady incompressible flow is extended to a case where the wing forms either an open or closed circular arc. The generalization is based on an integral formulation of the problem, which resembles the one used by Guermond (1990) for a plane curved wing. A second-order approximation is obtained for the load distribution on two model wings, one resembling that of a gliding parachute, and the other resembling a short duct.

1. Introduction

The asymptotic theory of high-aspect-ratio planar unswept wings in incompressible steady flow can be traced back to the pioneering words of Friderichs (1966) and Van Dyke (1975). Extensions of this theory for curved and swept wings were developed by Thurber (1965), Cheng (1978), Kida & Miyai (1978), Guermond (1990), and some others. We could not find any extension of this theory for arched (non-planar) wings, like those depicted in figures 1 and 2, although they are becoming quite common in aeronautical applications – for example, in gliding parachutes. Such an extension is the subject matter of this exposition.

Given a finite wing in incompressible steady flow, an asymptotic solution for the pressure distribution over the wing can be obtained in two basic ways. One is by the method of matched asymptotic expansions, as, for example, that used by Van Dyke (1975). The other is by an asymptotic expansion of a boundary integral equation, as, for example, that used by Guermond (1990). It seems that the latter approach leads to simpler derivations and therefore will be preferred over the former. The following theory will be conceptually based on Guermond's work, although we shall use a different boundary integral equation to avoid Hadamard-sense principal value integrals.

2. The model

Consider an infinitesimally thin wing positioned in a steady uniform flow of incompressible inviscid fluid.† An infinitesimally thin vortical wake is postulated to exist part the wing, starting at the trailing edge and extending to infinity. To avoid nonlinearity in formulation of the problem, the wake geometrical shape is assumed to be known *a priori*; specifically, it is assumed that vorticity is carried from the trailing edge of the wing along straight parallel lines, say, in the direction of the oncoming flow.

† Since the problem is steady, the results obtained below can be readily adapted to the subsonic case by exploiting the similarity rule of Göthert (e.g. Ashley & Landahl 1965, pp. 124–126).

Let U, ρ and b_0 be the flow velocity far form the wing, fluid density and maximal semi-chord of the wings. In subsequent derivations it will prove convenient to use dimensionless quantities, having b_0, U, $\frac{1}{2}\rho U^2$ and $b_0 U$ as units of length, velocity, pressure, and velocity potential respectively. Use of dimensionless quantities will be implicitly assumed hereafter.

Select a right-handed Cartesian coordinate system with unit basis vectors $\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z$, in such a way that the x-axis coincides with the direction of the oncoming flow (and, by assumption, with the direction of the wake past the wing).

For some real positive s, the generalized semi-span, let b and x_{TE} be the semi-chord of the wing and the x-coordinate of the wing's trailing edge, respectively, each continuous on $[-s,s]$ and analytic on $(-s,s)$.

Using spanwise and streamwise coordinates, the wing and wake surfaces can be defined on the rectangle $\Xi_{wing} = [-1,1]\times[-s,s]$ and the stripe $\Xi_{wake} = (1,\infty)\times[-s,s]$ by the vector-valued function $\boldsymbol{r}_s = x_s\boldsymbol{e}_x + y_s\boldsymbol{e}_y + z_s\boldsymbol{e}_z = x_s\boldsymbol{e}_x + \boldsymbol{r}_{yz}$, such that

$$\text{for each } \langle\xi_1,\xi_2\rangle\in[-1,\infty)\times[-s,s], \quad x_s(\xi_1,\xi_2) = x_{TE}(\xi_2) + b(\xi_2)(\xi_1 - 1), \tag{1}$$

$$\text{for each } \langle\xi_1,\xi_2\rangle\in(1,\infty)\times[-s,s], \quad \boldsymbol{r}_{yz}(\xi_1,\xi_2) = \boldsymbol{r}_{yz}(1,\xi_2), \tag{2}$$

$$\text{for each } \langle\xi_1,\xi_2\rangle\in[-1,1]\times[-s,s], \quad \boldsymbol{r}_{yz}(\xi_1,\xi_2) = \boldsymbol{r}_{yz}(1,\xi_2) + \delta\boldsymbol{r}_\delta(\xi_1,\xi_2). \tag{3}$$

In (3), δ is the maximal deviation of the wing's mean camber surface from that formed by the direct extension of the wake, and $\boldsymbol{r}_\delta \equiv y_\delta\boldsymbol{e}_y + z_\delta\boldsymbol{e}_z$; it is assumed that δ is small compared with unity, whereas each of $|\boldsymbol{r}_\delta|$, $|\partial\boldsymbol{r}_\delta/\partial\xi_1|$ and $s|\boldsymbol{r}_\delta/\partial\xi_2|$ is of the order unity on $(-1,1)\times(-s,s)$.

Let μ and p, each defined on $[-1,\infty)\times[-s,s]$, be the potential and pressure jumps across the combined wing/wake surface. These two jumps are formally related by

$$p = -2(\boldsymbol{e}_x + \boldsymbol{v}_s)\cdot\boldsymbol{\nabla}_2\mu = -2(\boldsymbol{e}_x + \boldsymbol{v}_s)\cdot\left\{\frac{\boldsymbol{n}}{|\boldsymbol{n}|^2}\times\frac{\partial(\mu,\boldsymbol{r}_s)}{\partial(\xi_1,\xi_2)}\right\}, \tag{4}$$

where $\boldsymbol{\nabla}_2$ is the surface gradient operator, $\boldsymbol{v}_s$ is the average of the perturbation velocities on the two sides of the wing's surface, and

$$\boldsymbol{n} = \partial\boldsymbol{r}_s/\partial\xi_1 \times \partial\boldsymbol{r}_s/\partial\xi_2 \tag{5}$$

is a local normal to the surface (see Baskin *et al.* 1976; also, Iosilevskii & Iosilevskii 1994).

As no pressure discontinuity can exist across the wake, equation (4) implies that μ is constant along streamlines associated with velocity $\boldsymbol{e}_x + \boldsymbol{v}_s$. Retrospectively, by assuming that the wake extends in the direction of the oncoming flow, we have neglected y- and z-components of $\boldsymbol{v}_s$ in the wake. In any event, under the present circumstances,

$$\text{for each } \langle\xi_1,\xi_2\rangle\in(1,\infty)\times[-s,s], \quad \mu(\xi_1,\xi_2) = \mu(1,\xi_2). \tag{6}$$

To ensure continuity of the potential in the immediate vicinity of the edges of the combined wing/wake surface, the value of the potential jump at the edges is assumed to be zero; specifically

$$\text{for each } \xi_2\in[-s,s], \quad \mu(-1,\xi_2) = 0, \tag{7}$$

$$\text{for each } \xi_1\in[-1,\infty), \quad \mu(\xi_1,-s) = \mu(\xi_1,s) = 0. \tag{8}$$

The last equation applies only in the case where the wing forms an open arc, i.e. when it has side edges. In the case where the wing forms a closed arc, i.e. when it resembles a duct or a ring, the zero on the right-hand side of (8) should be omitted.

From the Biot–Savart law it follows that for each $\langle\xi_1,\xi_2\rangle \in (-1,\infty)\times(-s,s)$,

$$v_s(\xi_1',\xi_2') = \frac{1}{4\pi}\int\!\!\!\!-\!\!\int_{\Xi_{wing}\cup\Xi_{wing}} \frac{\partial(\mu, r_s)}{\partial(\xi_1,\xi_2)} \times \frac{r_s(\xi_1',\xi_2')-r_s(\xi_1,\xi_2)}{|r_s(\xi_1',\xi_2')-r_s(\xi_1,\xi_2)|^3}\,d\xi_1\,d\xi_2, \tag{9}$$

where the bar across the integral sign indicates principal value in the Cauchy sense (see Baskin *et al.* 1976; also, Iosilevskii & Iosilevskii 1994). This equation, when used with the impermeability condition on the surface of the wing,

$$\text{for each } \langle\xi_1,\xi_2\rangle\in(-1,1)\times(-s,s),\quad [e_x + v_s(\xi_1,\xi_2)]\cdot n(\xi_1,\xi_2) = 0, \tag{10}$$

yields what is commonly known as the *boundary integral equation for* μ. The objective of this exposition is to find a solution of this equation, subject to (6), (7) and (8), for an arched wing. Such a wing will be specified in the next section. In the interim we proceed by simplifying equation (9) under assumptions already made.

Separate the wing and wake integration domains in equation (9). For the wake part, it follows from (1), (2) and (6) that for each $\langle\xi_1,\xi_2\rangle\in(1,\infty)\times(-s,s)$,

$$\frac{\partial(\mu,r_s)}{\partial(\xi_1,\xi_2)}\times\frac{\Delta r_s}{|\Delta r_s|^3} = -\frac{d\mu(1,\xi_2)}{d\xi_2}\frac{\partial x_s}{\partial\xi_1}\frac{e_x\times\Delta r_{yz}}{(|\Delta r_{yz}|^2+\Delta x_s^2)^{3/2}} = \frac{\partial}{\partial\xi_1}\left\{\frac{d\mu(1,\xi_2)}{d\xi_2}\frac{e_x\times\Delta r_{yz}}{|\Delta r_{yz}|^2}\frac{\Delta x_s}{|\Delta r_s|}\right\},$$

where

$$\Delta r_{yz} = r_{yz}(\xi_1',\xi_2') - r_{yz}(1,\xi_2),\quad \Delta x_s = x_s(\xi_1',\xi_2')-x_s(\xi_1,\xi_2),\quad \Delta r_s = r_s(\xi_1',\xi_2')-r_s(\xi_1,\xi_2).$$

Hence, one may readily integrate with respect to ξ_1 on $(1,\infty)$. Subject to (7), the integration yields

$$\begin{aligned} v_s(\xi_1',\xi_2') = {} & \frac{1}{4\pi}\int_{-1}^{1} d\xi_1 \int\!\!\!\!-_{-s}^{s} d\xi_2 \frac{\partial(\mu,r_s)}{\partial(\xi_1,\xi_2)}\times\frac{r_s(\xi_1',\xi_2')-r_s(\xi_1,\xi_2)}{|r_s(\xi_1',\xi_2')-r_s(\xi_1,\xi_2)|^3} \\ & - \frac{1}{4\pi}\int_{-s}^{s}\frac{d\mu(1,\xi_2)}{d\xi_2}\frac{e_x\times[r_{yz}(\xi_1',\xi_2')-r_{yz}(1,\xi_2)]}{|r_{yz}(\xi_1',\xi_2')-r_{yz}(1,\xi_2)|^2}\left\{1+\frac{x_s(\xi_1',\xi_2')-x_s(1,\xi_2)}{|r_s(\xi_1',\xi_2')-r_s(1,\xi_2)|}\right\}d\xi_2. \end{aligned} \tag{11}$$

Both integrals appearing on the right-hand side of (11) are regular with respect to δ. Thus, at the leading order with respect to δ, all occurrences of $r_{yz}(\xi_1',\xi_2')$ and $r_{yz}(\xi_1,\xi_2)$ can be replaced by $r_{yz}(1,\xi_2')$, and $r_{yz}(1,\xi_2)$, respectively (see Ashley & Landahl 1965). In this context, we note that since (11) presumes that the wake extends rectilinearly in the direction of the oncoming flow – which is a true description of the wake only when the flow remains unperturbed behind the wing, i.e. when $\delta = 0$ – equation (11) is *a priori* correct only to leading order with respect to δ.

3. Arched wing

The particular wing to be considered is shown in figure 1. It is assumed that the wing is so designed that, to within terms of order δ, the projection of its trailing edge on the (y,z)-plane is an arc of radius R and angle $2s$. It is understood that s takes on values in the interval $(0,\pi]$; limiting cases $s\to 0$ and $s=\pi$ correspond to the plane and ring-like (or duct-like) wings, respectively.

Under the present assumptions, convenient coordinates are cylindrical, $\langle r,\xi_2,x\rangle$, where ξ_2 is used as the polar angle measured from the z-axis (see figure 1). Thus, with

$$e_r(\xi_2) = e_y\sin\xi_2 + e_z\cos\xi_2 \quad\text{and}\quad e_\phi(\xi_2) = e_y\cos\xi_2 - e_z\sin\xi_2, \tag{12}$$

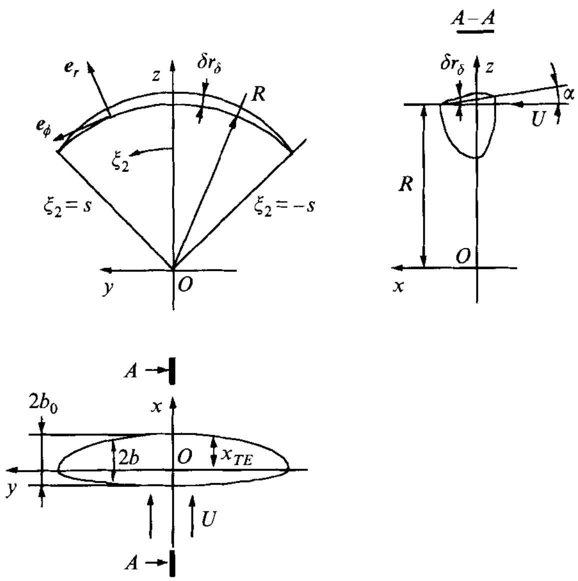

FIGURE 1. General arched wing.

the respective local unit basis vectors, equations (2) and (3) take the form

$$\text{for each } \langle \xi_1, \xi_2 \rangle \in [-1, 1] \times [-s, s], \quad \boldsymbol{r}_{yz}(\xi_1, \xi_2) = [R + \delta r_\delta(\xi_1, \xi_2)]\, \boldsymbol{e}_r(\xi_2), \tag{13}$$

$$\text{for each } \langle \xi_1, \xi_2 \rangle \in (1, \infty) \times [-s, s], \quad \boldsymbol{r}_{yz}(\xi_1, \xi_2) = R\boldsymbol{e}_r(\xi_2). \tag{14}$$

Subject to (12)–(14), equations (5) and (11) respectively become

$$\boldsymbol{n}(\xi_1', \xi_2') = [R + \delta r_\delta(\xi_1', \xi_2')] \left[\boldsymbol{e}_r(\xi_2')\, b(\xi_2') - \boldsymbol{e}_x \delta \frac{\partial r_\delta(\xi_1', \xi_2')}{\partial \xi_1'} \right] - \boldsymbol{e}_\phi(\xi_2') \delta \frac{\partial(x_s, r_\delta)}{\partial(\xi_1', \xi_2')} \tag{15}$$

and

$$\begin{aligned} \boldsymbol{v}_s(\xi_1', \xi_2') = {} & \frac{1}{8\pi R} \int_{-s}^{s} \mathrm{d}\xi_2 \frac{\mathrm{d}\mu(1, \xi_2)}{\mathrm{d}\xi_2} \{\boldsymbol{e}_\phi(\xi_2') - \boldsymbol{e}_r(\xi_2') \cot \tfrac{1}{2}(\xi_2' - \xi_2)\} \left\{ 1 + \frac{x_s(\xi_1', \xi_2') - x_s(1, \xi_2)}{|\Delta \boldsymbol{r}_{so}(\xi_1', \xi_2', 1, \xi_2)|} \right\} \\ & + \frac{R}{4\pi} \int_{-1}^{1} \mathrm{d}\xi_1 \int_{-s}^{s} \mathrm{d}\xi_2 \frac{\partial \mu}{\partial \xi_1} \{\boldsymbol{e}_\phi(\xi_2') \sin(\xi_2' - \xi_2) - \boldsymbol{e}_r(\xi_2') \cos(\xi_2' - \xi_2)\} \\ & \times \frac{x_s(\xi_1', \xi_2) - x_s(\xi_1, \xi_2)}{|\Delta \boldsymbol{r}_{so}(\xi_1', \xi_2', \xi_1, \xi_2)|^3} \\ & + \frac{R}{4\pi} \int_{-1}^{1} \mathrm{d}\xi_1 \int_{-s}^{s} \mathrm{d}\xi_2 \frac{\partial(\mu, x_s)}{\partial(\xi_1, \xi_2)} \frac{\boldsymbol{e}_r(\xi_2') \sin(\xi_2' - \xi_2) - \boldsymbol{e}_\phi(\xi_2')[1 - \cos(\xi_2' - \xi_2)]}{|\Delta \boldsymbol{r}_{so}(\xi_1', \xi_2', \xi_1, \xi_2)|^3} \\ & - \boldsymbol{e}_x \frac{R^2}{4\pi} \int_{-1}^{1} \mathrm{d}\xi_1 \int_{-s}^{s} \mathrm{d}\xi_2 \frac{\partial \mu}{\partial \xi_1} \frac{1 - \cos(\xi_2' - \xi_2)}{|\Delta \boldsymbol{r}_{so}(\xi_1', \xi_2', \xi_1, \xi_2)|^3} + \ldots, \end{aligned} \tag{16}$$

where

$$|\Delta \boldsymbol{r}_{so}(\xi_1', \xi_2', \xi_1, \xi_2)|^2 = 4R^2 \sin^2 \frac{\xi_2' - \xi_2}{2} + [x_s(\xi_1', \xi_2') - x_s(\xi_1, \xi_2)]^2, \tag{17}$$

$\langle \xi_1', \xi_2' \rangle \in (-1, 1) \times (-s, s)$, and the ellipsis stands for the terms which are an order higher with respect to δ than $\boldsymbol{v}_s$ itself (cf. the last paragraph of the preceding section). In deriving (16) we have used the relation

$$\partial x_s(\xi_1, \xi_2)/\partial \xi_1 = b(\xi_2), \tag{18}$$

immediately following from (1).

4. Asymptotic solution for an arched wing

We attempt to solve equation (10), subject to (7), (15) and (16), asymptotically, using the reciprocal $\epsilon = R^{-1}$ as a small parameter. Toward this end we limit possible wing configurations by the following assumptions:

(i) $|x_{TE}(\cdot) - x_{TE}(0)|$ is of the order unity on $[-s, s]$;
(ii) if the wing forms an *open* arc, then $s < \pi/2$, $Rs \gg 1$, and $b(\pm s) = 0$.

Note that under assumption (i) the wing is allowed to have a small sweep and to be positioned at a small sideslip angle relative to the flow.

For any three functions f, g, and h, analytic on (a, b), it is shown in the Appendix that if the interval (a, b) includes zero, and $h(0) \neq 0$, then

$$\int_a^b \frac{4f(\xi)\sin^2(\xi/2) + \epsilon g(\xi)}{[4\sin^2(\xi/2) + \epsilon^2 h(\xi)^2]^{3/2}} \mathrm{d}\xi = \frac{2g(0)}{\epsilon h^2(0)} - 2f(0)[\ln\epsilon + \ln|h(0)| + 1 - 3\ln 2]$$
$$+ f(b)\ln\tan(b/4) + f(a)\ln\tan(-a/4) - \int_a^b \operatorname{sign}(\xi)\ln|\tan(\xi/4)|\frac{\mathrm{d}f}{\mathrm{d}\xi}\mathrm{d}\xi + O(\epsilon\ln\epsilon),$$

whereas

$$\int_a^b \frac{\cot(\xi/2) f(\xi)\,\mathrm{d}\xi}{[4\sin^2(\xi/2) + \epsilon^2 g^2(\xi)]^{1/2}} = -4\left(\frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi}\right)_{\xi=0} \ln\epsilon + O(1).$$

Using these formulae, together with equations (7) and (8), equation (16) can be reduced to give the result

$$\boldsymbol{v}_s(\xi_1', \xi_2') = -\frac{\boldsymbol{e}_r(\xi_2')}{2\pi b(\xi_2')}\int_{-1}^{1} \frac{\partial\mu(\xi_1, \xi_2')}{\partial\xi_1}\frac{\mathrm{d}\xi_1}{\xi_1' - \xi_1} - \epsilon\frac{\boldsymbol{e}_r(\xi_2')}{8\pi}\int_{-s}^{s} \frac{\mathrm{d}\mu(1, \xi_2)}{\mathrm{d}\xi_2}\cot\tfrac{1}{2}(\xi_2' - \xi_2)\,\mathrm{d}\xi_2$$
$$+ \epsilon\ln\epsilon\frac{\boldsymbol{e}_x}{4\pi}\mu(1, \xi_2') + \epsilon\frac{\boldsymbol{e}_x}{4\pi}\int_{-1}^{1} \frac{\partial\mu(\xi_1, \xi_2')}{\partial\xi_1}\ln|\xi_1' - \xi_1|\mathrm{d}\xi_1 + \epsilon\frac{\boldsymbol{e}_x}{4\pi}\mu(1, \xi_2')[1 - 3\ln 2 + \ln b(\xi_2')]$$
$$+ \epsilon\frac{\boldsymbol{e}_x}{8\pi}\int_{-s}^{s} \operatorname{sign}(\xi_2' - \xi_2)\ln(\tan\tfrac{1}{4}|\xi_2' - \xi_2|)\frac{\partial\mu(1, \xi_2)}{\partial\xi_2}\mathrm{d}\xi_2 + \ldots, \tag{19}$$

where the ellipsis stands for higher-order terms both with respect to ϵ and with respect to δ.

Assume, subject to *a posteriori* verification, that μ can be expanded into asymptotic series of the form

$$\mu = \mu_0 + \epsilon\ln\epsilon\mu_1 + \epsilon\mu_2 + (\epsilon\ln\epsilon)^2\mu_3 + \ldots, \tag{20}$$

where each $\mu_0, \mu_1, \mu_2, \ldots$ is defined on $[-1, 1] \times [-s, s]$, and satisfies (7) and (8). Substitute (20) in (19); then substitute the resulting expression, together with (15), in (10); finally, equate to zero the multipliers of ϵ^0, $\epsilon\ln\epsilon, \epsilon^1, \ldots$. The first equation thereby obtained is

$$\frac{1}{2\pi}\int_{-1}^{1} \frac{\partial\mu_0(\xi_1, \xi_2')}{\partial\xi_1}\frac{\mathrm{d}\xi_1}{\xi_1' - \xi_1} = -\delta\frac{\partial r_\delta(\xi_1', \xi_2')}{\partial\xi_1'} + O(\delta^2); \tag{21}$$

it implies that $\mu_0 = O(\delta)$. Accordingly, to remain consistent with the preceding derivations – which are correct only to leading order with respect to δ – we are bounded to neglect all contributions to μ_0, μ_1, μ_2, ... which are of order δ^2. As a consequence, two subsequent equations take the form

$$\mu_1(\xi_1', \xi_2') = O(\delta^2), \tag{22}$$

$$\frac{1}{2\pi}\int_{-1}^{1} \frac{\partial\mu_2(\xi_1, \xi_2')}{\partial\xi_1}\frac{d\xi_1}{\xi_1'-\xi_1} = -\frac{1}{8\pi}b(\xi_2')\int_{-s}^{s} \frac{d\mu_0(1, \xi_2)}{d\xi_2}\cot\frac{\xi_2'-\xi_2}{2}d\xi_2 + O(\delta^2), \tag{23}$$

In (21)–(23), $\langle\xi_1', \xi_2'\rangle \in (-1, 1)\times(-s, s)$.

Equations (21)–(23) are similar, *mutatis mutandis*, to those obtained by Van Dyke (1975) for straight planar wings; they become identical in the limit where $s \rightarrow 0$. The effect of arching is explicit in the integral kernel on the right-hand side of (23), and implicit in the geometrical angle of attack on the right-hand side of (21). Equation (22) is noteworthy since μ_1 is known to be of order δ (rather than δ^2) for a comparably curved planar wing (see Thurber 1965; also, Guermond 1990).

By analogy with the planar case, equations (21)–(23) imply that Prandtl's lifting-line theory (Prandtl, Wiesselsberger & Betz 1921) can be adopted for arched wings – with two modifications. One is that the wing (either in symmetric or asymmetric flight) should be presented by a *planar* arched vortex positioned perpendicular to the direction in which the wake extends (in the present notation, the arc should be in the (y, z)-plane); the other is that the (infinite) velocity induced by this vortex on itself should be disregarded.

By prescribing that the derivative $\partial\mu/\partial\xi_1$ should vanish at the trailing edge and have an integrable singularity at the leading edge, integral equations (21) and (23) can be inverted (see Söhngen 1939; also Guermond, 1990) to obtain

$$\frac{\partial\mu_0(\xi_1, \xi_2')}{\partial\xi_1} = \frac{2\delta}{\pi}\left(\frac{1-\xi_1}{1+\xi_1}\right)^{1/2}\int_{-1}^{1}\left(\frac{1+\xi_1'}{1-\xi_1'}\right)^{1/2}\frac{\partial r_\delta(\xi_1', \xi_2')}{\partial\xi_1'}\frac{d\xi_1'}{\xi_1-\xi_1'} + O(\delta^2), \tag{24}$$

$$\frac{\partial\mu_2(\xi_1, \xi_2')}{\partial\xi_1} = -\frac{b(\xi_2')}{4\pi}\left(\frac{1-\xi_1}{1+\xi_1}\right)^{1/2}\int_{-s}^{s}\frac{d\mu_0(1, \xi_2)}{d\xi_2}\cot\frac{\xi_2'-\xi_2}{2}d\xi_2 + O(\delta^2), \tag{25}$$

with $\langle\xi_1, \xi_2'\rangle \in (-1, 1)\times(-s, s)$. In deriving (25) we have used the fact that the right-hand side of (23) is independent of ξ_1'.

5. Lift and drag coefficients of an arched wing

In the present notation, the lift coefficient C_L of the wing can be defined by

$$C_L = -\frac{1}{S}\int_{-1}^{1} d\xi_1 \int_{-s}^{s} d\xi_2\, p(\xi_1, \xi_2)\frac{\partial(x_s, y_s)}{\partial(\xi_1, \xi_2)}, \tag{26}$$

where S is an arbitrary reference area. The pressure jump p appearing in the integrand can be found straightforwardly from (4), (16), (20) and (22); thus

$$p(\xi_1, \xi_2) = -\frac{2}{b(\xi_2)}\left[\frac{\partial\mu_0(\xi_1, \xi_2)}{\partial\xi_1} + \epsilon\frac{\partial\mu_2(\xi_1, \xi_2)}{\partial\xi_1} + \dots\right], \tag{27}$$

with subsequent terms of order δ^2 and $(\epsilon\ln\epsilon)^2\delta$. At the same time, from (1), (12) and (13),

$$\frac{\partial(x_s, y_s)}{\partial(\xi_1, \xi_2)} = b(\xi_2)R\cos\xi_2 + O(\delta). \tag{28}$$

Hence, subject to (7),

$$C_L = \frac{2R}{S}\int_{-s}^{s} d\xi_2 \cos\xi_2 [\mu_0(1,\xi_2)+\epsilon\mu_2(1,\xi_2)+\ldots], \tag{29}$$

where

$$\mu_0(1,\xi_2') = \int_{-1}^{1} \frac{\partial \mu_0(\xi_1,\xi_2')}{\partial \xi_1} d\xi_1 = -2\delta\int_{-1}^{1}\left(\frac{1+\xi_1}{1-\xi_1}\right)^{1/2}\frac{\partial r_\delta(\xi_1,\xi_2')}{\partial \xi_1} d\xi_1 + O(\delta^2), \tag{30}$$

$$\mu_2(1,\xi_2') = \int_{-1}^{1} \frac{\partial \mu_2(\xi_1,\xi_2')}{\partial \xi_1} d\xi_1 = -\tfrac{1}{4}b(\xi_2')\int_{-s}^{s}\frac{d\mu_0(1,\xi_2)}{d\xi_2}\cot\frac{\xi_2'-\xi_2}{2} d\xi_2 + O(\delta^2), \tag{31}$$

by (24) and (25).

The induced drag coefficient C_D of the wing is most conveniently found from energy considerations (see, for example, Ashley & Landahl 1965, pp. 135–136). To avoid repetition, we proceed directly from equation (7.41) of the last reference, which holds for both planar and non-planar wakes. Thus, in the present notation,

$$C_D = -\frac{1}{S}\lim_{\xi_1\to\infty}\int_{-s}^{s}\mu(\xi_1,\xi_2)\,\boldsymbol{v}_s(\xi_1,\xi_2)\cdot\left[\boldsymbol{e}_x\times\frac{\partial \boldsymbol{r}_{yz}(\xi_1,\xi_2)}{\partial \xi_2}\right] d\xi_2. \tag{32}$$

With (6) and (13)–(15), equation (32) becomes

$$C_D = \frac{1}{4\pi S}\int_{-s}^{s} d\xi_2\,\mu(1,\xi_2)\int_{-s}^{s} d\xi_2'\frac{d\mu(1,\xi_2')}{d\xi_2'}\cot\frac{\xi_2-\xi_2'}{2}. \tag{33}$$

As could be expected, in the limit where $s\to 0$, (33) reduces to the well-known expression for planar wings (see Ashley & Landahl 1965, p. 136).

Substitute (20) in (33). For any m and n in $\{0,1,2,\ldots\}$,

$$\int_{-s}^{s} d\xi_2\,\mu_n(1,\xi_2)\int_{-s}^{s} d\xi_2'\frac{d\mu_m(1,\xi_2')}{d\xi_2'}\cot\frac{\xi_2-\xi_2'}{2} = \int_{-s}^{s} d\xi_2\,\mu_m(1,\xi_2)\int_{-s}^{s} d\xi_2'\frac{d\mu_n(1,\xi_2')}{d\xi_2'}\cot\frac{\xi_2-\xi_2'}{2}\ldots. \tag{34}$$

This is easily proved: integrate by parts with respect to ξ_2 on both sides of (34); then since each μ_n and μ_m satisfies (8), the identity of the resulting expressions becomes immediately apparent.

Hence, with (22) and (31), asymptotic series for the drag coefficient can be reduced to the form

$$C_D = -\frac{1}{\pi S}\int_{-s}^{s}\frac{\mu_0(1,\xi_2)\mu_2(1,\xi_2)}{b(\xi_2)} d\xi_2 - \frac{2\epsilon}{\pi S}\int_{-s}^{s}\frac{\mu_2^2(1,\xi_2)}{b(\xi_2)} d\xi_2 + \ldots. \tag{35}$$

6. A pseudo-elliptic arched wing

As an example, consider a non-cambered geometrically untwisted parachute wing with (pseudo-elliptic) chord distribution

$$\text{for each } \xi_2\in[-s,s],\quad b(\xi_2) = (\tan^2(\tfrac{1}{2}s)-\tan^2(\tfrac{1}{2}\xi_2))^{1/2}(\cos\xi_2\tan(\tfrac{1}{2}s))^{-1}. \tag{36}$$

This wing will be assumed to execute a symmetric flight with mid-section positioned at angle of attack α, in which case

$$\text{for each } \langle\xi_1,\xi_2\rangle\in(1,1)\times(-s,s),\quad \delta\,\partial r_\delta(\xi_1,\xi_2)/\partial\xi_1 = -\alpha b(\xi_2)\cos\xi_2. \tag{37}$$

Note that when $s \to 0$ the problem reduces to an elliptic untwisted non-cambered wing in symmetric flight. Also note the curvature-associated wing twist at $\alpha > 0$.

Since the right-hand side of (37) is independent of ξ_1, integration in equation (30) is immediate; the result is

$$\mu_0(1, \xi_2) = 2\pi\alpha b(\xi_2) \cos \xi_2. \tag{38}$$

Consequent integrations in equations (31), (29) and (35) are fairly straightforward, although lengthy; the final results are

$$\mu_2(1, \xi_2) = -\pi^2 \alpha b(\xi_2)(2 \tan(\tfrac{1}{2}s) \cos^2(\tfrac{1}{2}\xi_2))^{-1} + O(\alpha^2), \tag{39}$$

$$C_L = \pi^2 \alpha S^{-1}(8R \cos(\tfrac{1}{2}s) \tan(\tfrac{1}{4}s) - \pi + \ldots), \tag{40}$$

$$C_D = \pi^3 \alpha^2 S^{-1}\{1 - 4\pi R^{-1} \cot^3(\tfrac{1}{2}s)[1 - (1 - \tan^2(\tfrac{1}{2}s))^{1/2} - \tfrac{3}{8}\tan^2(\tfrac{1}{2}s) + \tfrac{1}{32}\tan^4(\tfrac{1}{2}s)] + \ldots\}. \tag{41}$$

The last two equations can be brought into a more familiar form with

$$S = 2R \int_{-s}^{s} b(\xi_2) \cos \xi_2 \, d\xi_2 = 4\pi R \tan(\tfrac{1}{4}s) \tag{42}$$

the area, and

$$A = 4R^2 S^{-1} \sin^2 s = R \sin^2 s (\pi \tan(\tfrac{1}{4}s))^{-1} \tag{43}$$

the aspect ratio of the wing's projection on the (x, y)-plane. Thus, (40) and (41), respectively become

$$C_L = 2\pi\alpha \cos(\tfrac{1}{2}s)(1 - 2A^{-1} \cos(\tfrac{1}{2}s) \cos^4(\tfrac{1}{4}s) + \ldots), \tag{44}$$

$$C_D = \frac{C_L^2}{\pi A} \cos^4(\tfrac{1}{4}s)\{1 + 4A^{-1} \cos^2(\tfrac{1}{4}s) \cos(\tfrac{1}{2}s)(\cos^4(\tfrac{1}{4}s) + 3\cos^2(\tfrac{1}{2}s))$$
$$- 16A^{-1} \cos^4(\tfrac{1}{2}s) \cot(\tfrac{1}{4}s) \cot(\tfrac{1}{2}s)[1 - (1 - \tan^2(\tfrac{1}{2}s))^{1/2}] + \ldots\}. \tag{45}$$

Note that in the limit where $s \to 0$, (44) and (45) reduce to the well-known classical results for high-aspect ratio straight elliptical wings (Van Dyke 1975, p. 168; Ashley & Landahl 1965, p. 137). Also note that the sectional circulation of the wing, as given by the value of the potential jump at the trailing edge, is

$$\mu(1, \xi_2) = 2\pi\alpha b(\xi_2) \cos \xi_2 [1 - 2A^{-1} \cos^2(\tfrac{1}{4}s) \cos^3(\tfrac{1}{2}s)(\cos^2(\tfrac{1}{2}\xi_2) \cos \xi_2)^{-1} + \ldots], \tag{46}$$

by (20), (22), (38), (39) and (44). Essentially, equation (46) describes an elliptic distribution, somewhat weakened at the tips owing to both the intrinsic 'washout' (37) and the reduced effect of tip vortices at the mid-wing.

7. A lifting cylindrical ring

As another useful example, consider a straight cylindrical ring of constant chord (in which case $b = 1$, by definition) positioned almost perpendicular to the flow (see figure 2). Let α be the angle between the normal to the plane of the ring and the x-axis. To satisfy assumption (i) of §4, the angle α will be assumed sufficiently small, so that $R\alpha$ is of the order of unity.

Under these circumstances, equations (37) and (38) hold. Consequent integration in (31) yields

$$\mu_2(1, \xi_2) = -\pi^2 \alpha \cos \xi_2 + O(\alpha^2). \tag{47}$$

Accordingly, from (20),

$$\mu(1, \xi_2) = 2\pi\alpha \cos \xi_2 (1 - \tfrac{1}{2}\pi R^{-1} + \ldots). \tag{48}$$

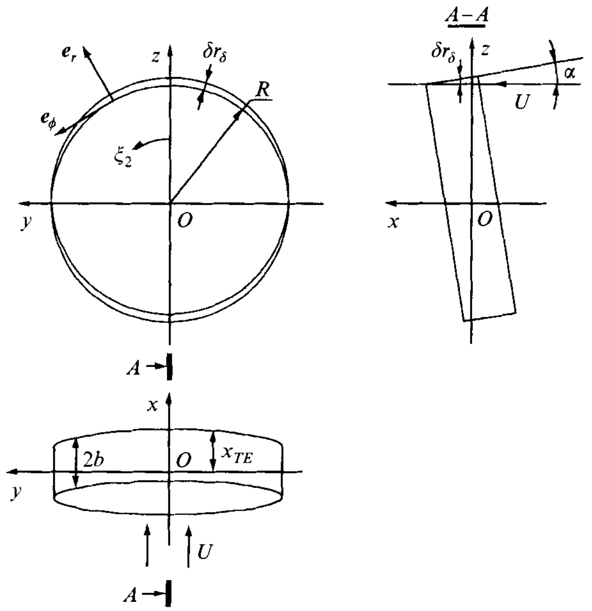

FIGURE 2. A lifting ring.

In other words, the sectional circulation is proportional to the local angle of attack.

To obtain the lift and drag coefficients, substitute (38) and (47) in (29) and (35). Based on the projection area $S = 4R$ of the wing, the results are

$$C_L = 4\pi^2\alpha R S^{-1}(1-\tfrac{1}{2}\pi R^{-1}+\dots) = \pi^2\alpha(1-\tfrac{1}{2}\pi R^{-1}+\dots), \tag{49}$$

$$C_D = 2\pi^3\alpha^2 S^{-1}(1-\pi R^{-1}+\dots) = \frac{1}{2\pi}C_L^2 R^{-1}[1+O(R^{-2}\ln R)]. \tag{50}$$

Both expressions agree with the comparable results of Belotserkovskii (1967) – see equation (4.34), subject to (4.25), (1.8) and (1.9), and the first paragraph on p. 133 of that reference.

Appendix. Asymptotic expansions of the typical integrals

Let (a,b) be an open interval containing zero, and ϵ be a small positive parameter. Also, let, for each ξ in (a,b),

$$F(\xi,\epsilon) = \frac{4f(\xi)\sin^2(\xi/2)+\epsilon g(\xi)}{[4\sin^2(\xi/2)+\epsilon^2 h(\xi)^2]^{3/2}}, \tag{A 1}$$

where f, g and h are analytic on (a,b). It will be assumed that $h(0) \neq 0$. We seek an asymptotic expansion with respect to ϵ of the integral

$$I(\epsilon) = \int_a^b F(\xi,\epsilon)\,\mathrm{d}\xi. \tag{A 2}$$

A generic expansion of this type was studied by Guermond (1988, 1990). In either of the two references cited one can find a general paradigm to obtain any term in the expansion. However, with (A 1), in our opinion the first terms of the expansion can be found more easily by a direct approach which is described below.

Let λ be an auxiliary small parameter, such that $\epsilon \ll \lambda \ll 1$. Divide the interval (a, b) of integration into three parts $(a, -\lambda), (-\lambda, \lambda)$ and (λ, b), i.e.

$$I(\epsilon) = \int_{-\lambda}^{\lambda} F(\xi, \epsilon)\,\mathrm{d}\xi + \int_{a}^{-\lambda} F(\xi, \epsilon)\,\mathrm{d}\xi + \int_{\lambda}^{b} F(\xi, \epsilon)\,\mathrm{d}\xi. \quad \text{(A 3)}$$

In the first (singular) term on the right-hand side of (A 3), change the variable of integration to $\zeta = \xi/\epsilon$ so as to obtain

$$\int_{-\lambda}^{\lambda} F(\xi, \epsilon)\,\mathrm{d}\xi = \frac{1}{\epsilon}\int_{-\lambda/\epsilon}^{\lambda/\epsilon} \frac{g(\zeta\epsilon) + \epsilon f(\zeta\epsilon)\,(4/\epsilon^2)\sin^2(\frac{1}{2}\epsilon\zeta)}{[(4/\epsilon^2)\sin^2(\frac{1}{2}\epsilon\zeta) + h^2(\zeta\epsilon)]^{3/2}}\,\mathrm{d}\zeta. \quad \text{(A 4)}$$

Note that

$$\lim_{\epsilon \to 0} \frac{2}{\epsilon}\sin\left(\frac{\epsilon\zeta}{2}\right) = \zeta. \quad \text{(A 5)}$$

Expand the integrand in (A 4) in a Maclaurin series with respect to ϵ. Since all integrals involving odd powers of ζ will vanish by symmetry considerations, such an expansion yields

$$\int_{-\lambda}^{\lambda} F(\xi, \epsilon)\,\mathrm{d}\xi = \frac{1}{\epsilon}\int_{-\lambda/\epsilon}^{\lambda/\epsilon} \frac{g(0) + \epsilon f(0)\,\zeta^2}{[\zeta^2 + h^2(0)]^{3/2}}\,\mathrm{d}\zeta + O(\epsilon \ln \epsilon); \quad \text{(A 6)}$$

whence, upon integration,

$$\int_{-\lambda}^{\lambda} F(\xi, \epsilon)\,\mathrm{d}\xi = \frac{2g(0)}{\epsilon h^2(0)} - 2f(0)\,(1 + \ln \epsilon + \ln |h(0)| - \ln \lambda - \ln 2) + O(\epsilon \ln \epsilon). \quad \text{(A 7)}$$

We proceed now with the second and third integrals on the right-hand side of (A 3). Since F can be straightforwardly expanded in a Maclaurin series with respect to ϵ both on $(a, -\lambda)$ and on (λ, b), it immediately follows that

$$\begin{aligned}\int_{a}^{-\lambda} F(\xi, \epsilon)\,\mathrm{d}\xi + \int_{\lambda}^{b} F(\xi, \epsilon)\,\mathrm{d}\xi &= \int_{a}^{-\lambda} F(\xi, 0)\,\mathrm{d}\xi + \int_{\lambda}^{b} F(\xi, 0)\,\mathrm{d}\xi + O(\epsilon) \\ &= \int_{\lambda}^{-a} \frac{f(-\xi)\,\mathrm{d}\xi}{2\sin(\xi/2)} + \int_{\lambda}^{b} \frac{f(\xi)\,\mathrm{d}\xi}{2\sin(\xi/2)} + O(\epsilon). \quad \text{(A 8)}\end{aligned}$$

Further, integrate the right-hand side of (A 8) by parts to obtain

$$\begin{aligned}\int_{\lambda}^{-a} \frac{f(-\xi)\,\mathrm{d}\xi}{2\sin(\xi/2)} + \int_{\lambda}^{b} \frac{f(\xi)\,\mathrm{d}\xi}{2\sin(\xi/2)} &= f(b)\ln(\tan\tfrac{1}{4}b) + f(a)\ln(\tan\tfrac{1}{4}|a|) - 2f(0)\,(\ln \lambda - 2\ln 2) \\ &\quad - \int_{a}^{b} \operatorname{sign}(\xi)\,(\ln\tan\tfrac{1}{4}|\xi|)\frac{\mathrm{d}f}{\mathrm{d}\xi}\,\mathrm{d}\xi + O(\lambda \ln \lambda). \quad \text{(A 9)}\end{aligned}$$

Substitute (A 7)–(A 9) in (A 3). Noting that $I(\epsilon)$ should eventually be independent of λ, the final result takes on the form

$$\begin{aligned}\int_{a}^{b} \frac{4f(\xi)\sin^2(\xi/2) + \epsilon g(\xi)}{[4\sin^2(\xi/2) + \epsilon^2 h(\xi)^2]^{3/2}}\,\mathrm{d}\xi &= \frac{2g(0)}{\epsilon h^2(0)} - 2f(0)\,[\ln \epsilon + \ln |h(0)| + 1 - 3\ln 2] \\ &\quad + f(b)\ln(\tan\tfrac{1}{4}b) + f(a)\ln(\tan\tfrac{1}{4}|a|) - \int_{a}^{b} \operatorname{sign}(\xi)\,(\ln\tan\tfrac{1}{4}|\xi|)\frac{\mathrm{d}f}{\mathrm{d}\xi}\,\mathrm{d}\xi + O(\epsilon \ln \epsilon). \quad \text{(A 10)}\end{aligned}$$

By following the same steps as those leading to (A 7) and (A 8), it is easy to verify that

$$\int_a^b \frac{g(\xi)\,d\xi}{2\sin(\xi/2)\,[4\sin^2(\xi/2)+\epsilon^2 g^2(\xi)]^{1/2}} = -2\ln\epsilon\left(\frac{dg(\xi)}{d\xi}\right)_{\xi=0} + O(1). \qquad (A\ 11)$$

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